# The first-rank tensor field coupled to an electromagnetic field



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# The first-rank tensor field coupled to an electromagnetic field

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**Abstract.** The first-rank tensor field coupled to an external electromagnetic field is reduced to a constrained mechanical model. By using the constrained mechanical model we trace the origin of both the non-positive definiteness of the commutators and the non-causal modes of propagation. We find that these diseases have a common origin inherent in the constrained dynamical systems.

## 1. Introduction

The occurrence of acausal propagation as well as indefiniteness of the anticommutator in various relativistic wave equations has been studied almost from the beginning of relativistic field theory [1]. These diseases are by no means confined to higher-spin fields [2]. Since the same indefinite factor is present in the anticommutator as well as in the characteristic determinant, it has been conjectured for some time that both diseases have a common origin that can be traced to the occurrence of constraints. In fact, it was found [3] that constraints can cause the $c$-number Hamiltonian for such theories to become non-local. In Diract terminology, this implies that the Lagrange-multiplier fields have non-local behaviour and that the constraint matrices may become singular [4]. We have now recognised that these diseases are inherent in constrained systems and can be traced to the invertibility condition [5].

In this paper we study both a spin-0 and a spin-1 first-rank tensor field coupled to an electromagnetic field minimally, as well as directly, via Pauli terms. This theory can be reduced to a mechanical model with constraints and, by an appropriate choice of the Pauli term, can be made to display both the indefiniteness of the commutator as well as acausal propagation. Although indefiniteness of the commutator is not a defect, since it can always be remedied by interchanging annihilation and creation operators, the model allows us to show that both diseases have a common origin.

## 2. The first-rank tensor field

The Lagrange density for the first-rank tensor field of spin zero coupled to an electromagnetic field can be written

$$\mathscr{L} = \phi^{*\mu}[\Lambda_{\mu\nu}(D) + \Lambda^{(P)}_{\mu\nu}]\phi^{\nu} \tag{2.1}$$

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where

$$\Lambda_{\mu\nu}(D) = -(a_1 D^2 + a_2 m^2) g_{\mu\nu} - p_1 D_\mu D_\nu \tag{2.2}$$

$$\Lambda^{(P)}_{\mu\nu} = -\mathrm{i}e(1-\alpha_P) p_1 F_{\mu\nu}. \tag{2.3}$$

The term $\Lambda^{(P)}_{\mu\nu}$ is a 'Pauli' term and the minimal coupling has been introduced through the terms

$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu. \tag{2.4}$$

The parameters $a_1$, $a_2$ and $p_1$ are real. It is also useful to note that

$$[D_\mu, D_\nu] = -\mathrm{i}eF_{\mu\nu} = -\mathrm{i}e(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{2.5}$$

We now split the vector field $\phi^\mu$ into its irreducible spin parts according to

$$\phi_i^{(1)} = g_{i\alpha}\phi^\alpha = \phi_i \tag{2.6}$$

$$\phi^{(0)} = g_{0\alpha}\phi^\alpha = \phi_0. \tag{2.7}$$

In terms of this decomposition the Lagrangian becomes

$$\mathscr{L} = \dot{\phi}^*_\mu m_{\mu\nu} \dot{\phi}_\nu + (\dot{\phi}^*_\mu c_{\mu\nu} \phi_\nu - \phi^*_\mu \bar{c}_{\mu\nu} \dot{\phi}_\nu) - \phi^*_\mu r_{\mu\nu} \phi_\nu. \tag{2.8}$$

We have introduced the following notation:

$$\phi_\mu = (\phi_0, \phi_k) \equiv (\phi^{(0)}, \phi_k^{(1)}) \tag{2.9}$$

$$m_{\mu\nu} = m_\mu g_{\mu\nu} \qquad m_\mu = (m_0, m_k) \tag{2.10}$$

with

$$m_0 = a_1 + p_1 \qquad m_k = a_1. \tag{2.11}$$

Furthermore, with this splitting we have

$$c_{\mu\nu} \equiv [c^{(0)}, \tilde{c}_i^{(1,0)}, c_k^{(0,1)}, c_{ik}^{(1)}] \tag{2.12}$$

where

$$\begin{aligned} c^{(0)} &= -\mathrm{i}eA_0 m_0 & \tilde{c}_i^{(1,0)} &= -(1-\alpha_P)p_1 D_i \\ c_k^{(0,1)} &= -\alpha_P p_1 D_k & c_{ik}^{(1)} &= -\mathrm{i}eA_0 a_1 g_{ik}. \end{aligned} \tag{2.13}$$

The corresponding terms for $\bar{c}_{\mu\nu}$ are given by

$$\bar{c}_{\mu\nu} \equiv [c^{(0)}, c_i^{(1,0)}, \tilde{c}_k^{(0,1)}, c_{ik}^{(1)}] \tag{2.14}$$

and

$$c_i^{(1,0)} \equiv -\alpha_P p_1 D_i \qquad \tilde{c}_k^{(0,1)} = -(1-\alpha_P)p_1 D_k. \tag{2.15}$$

The $r_{\mu\nu}$ terms are

$$r_{\mu\nu} \equiv [R^{(0)}, R_i^{(1,0)}, R_k^{(0,1)}, R_{ik}^{(1)}] \tag{2.16}$$

$$R^{(0)} = -a_1\Delta_D + a_2 m^2 - e^2 A_0 m_0 \qquad \Delta_D = D_i D_i \tag{2.17}$$

$$R_i^{(1,0)} = \mathrm{i}ep_1[(1-\alpha_P)A_0 D_i + \alpha_P D_i A_0] \tag{2.18}$$

$$R_k^{(0,1)} = \mathrm{i}ep_1[\alpha_P A_0 D_k + (1-\alpha_P) D_k A_0] \tag{2.19}$$

$$R_{ik}^{(1)} = -[a_1(\Delta_D + e^2 A_0^2) - a_2 m^2] g_{ik} + p_1[\alpha_P D_i D_k + (1-\alpha_P) D_k D_i]. \tag{2.20}$$

In this form the spin content of the fields as well as their canonical structure is clearly exhibited in this decomposition.

We now define

$$\chi = (a_1 + p_1)/a_2 \tag{2.21}$$

where

$$a_2 = p \neq 0. \tag{2.22}$$

If we assume the existence of a Klein–Gordon divisor, then $\chi$ can assume only one of two values (see appendix 1 for a detailed discussion):

$$(a) \qquad \chi = 0 \qquad \text{with } a_1 = p, p_1 = -p \tag{2.23}$$

$$(b) \qquad \chi = 1 \qquad \text{with } a_1 = 0, p_1 = p. \tag{2.24}$$

Case $(a)$ corresponds to a Proca field (spin-1) and $\phi_\mu$ satisfies the Lorentz condition. Case $(b)$ corresponds to $\phi_\mu$ satisfying the Bianchi identity, describing a massive spin-0 particle. In the next section we develop a mechanical model which reduces to the above special cases for particular choices of parameters.

## 3. The mechanical model

A mechanical model with $2(N+1)$ degrees of freedom and the same constraint structure as the field theory discussed in the previous section is given by the Lagrangian

$$L = \dot{\phi}_\mu^* m_{\mu\nu} \dot{\phi}_\nu + (\dot{\phi}_\mu^* c_{\mu\nu} \phi_\nu - \phi_\mu^* \bar{c}_{\mu\nu} \dot{\phi}_\nu) - \phi_\mu^* r_{\mu\nu} \phi_\nu - V(\phi_\mu^*, \phi_\nu). \tag{3.1}$$

We assume summation over repeated indices. Here $\phi_\mu^*$, $\phi_\nu$ ($\mu$, $\nu = 0, 1, 2, \ldots, N$) are the coordinates and the potential term $V$ contains quadratic as well as higher-order terms in the coordinates. The 'kinetic mass matrix' $m_{\mu\nu}$ as well as the matrix $r_{\mu\nu}$ are both Hermitian. The matrix $c_{\mu\nu}$ is anti-Hermitian.

Since the matrix $m$ is Hermitian, it can always be diagonalised by a unitary transformation. We assume that this has been done and that

$$m_{\mu\nu} = m_\mu \delta_{\mu\nu} \tag{3.2}$$

with no summation over the repeated index of the eigenvalue $m_\mu$ from now on. This eigenvalue is called the kinetic mass [6].

The potential term plays no role in the constraint structure and is therefore dropped from now on. Dropping the potential term $V$, the Lagrangian can be written [7]

$$L = m_\mu \dot{\phi}_\mu^* \delta_{\mu\nu} \dot{\phi}_\nu + (\dot{\phi}_\mu^* c_{\mu\nu} \phi_\nu - \phi_\mu \bar{c}_{\mu\nu} \dot{\phi}_\nu) - \phi_\mu^* r_{\mu\nu} \phi_\nu. \tag{3.3}$$

We now assume that the mass matrix is singular of rank $r = N$ or 1 in order to analyse in detail the problem inherent in the constrained systems. The problem is to show that both the non-positive definiteness of the commutators and the non-causal modes of propagation originate from a lack of invertibility of the operators $O_d$ defined by equations (3.14) and (3.36). By a suitable choice of the unitary transformation that diagonalised $m$ we can arrange to have

$$(a) \qquad \begin{array}{ll} m_\mu = 0 & \text{for } \mu = 0 \\ m_\mu \neq 0 & \text{for } \mu = i = 1, 2, \ldots, N \end{array} \tag{3.4}$$

for $r = N$ and

$$(b) \qquad \begin{array}{ll} m_\mu \neq 0 & \text{for } \mu = 0 \\ m_\mu = 0 & \text{for } \mu = i = 1, 2, \ldots, N \end{array} \tag{3.5}$$

for $r = 1$. We adhere to this notation of letting, in both cases, Latin indices from the middle of the alphabet, namely $i, j, k, \ldots$, run from 1 to $N$.

The Euler-Lagrange equations corresponding to the Lagrangian (3.3) are

$$m_\mu \ddot{\phi}_\mu + d_{\mu\nu}\dot{\phi}_\nu + (\dot{c}_{\mu\nu} + r_{\mu\nu})\phi_\nu = 0 \tag{3.6}$$

$$\ddot{\phi}^*_\nu m_\nu - \dot{\phi}^*_\mu d_{\mu\nu} + \phi^*_\mu(-\dot{\bar{c}}_{\mu\nu} + r_{\mu\nu}) = 0 \tag{3.7}$$

where $d_{\mu\nu}$ is defined by

$$d_{\mu\nu} = c_{\mu\nu} + \bar{c}_{\mu\nu}. \tag{3.8}$$

We are now ready to analyse the constraint system corresponding to the Lagrangian (3.3).

### 3.1. Case (a)

The Euler-Lagrange equations (3.6) now split into equations of motion:

$$m_i \ddot{\phi}_i + d_{i\nu}\dot{\phi}_\nu + (\dot{c}_{i\nu} + r_{i\nu})\phi_\nu = 0 \tag{3.9}$$

and equations of constraint:

$$d_{0\nu}\dot{\phi}_\nu + (\dot{c}_{0\nu} + r_{0\nu})\phi_\nu = 0. \tag{3.10}$$

Taking the time derivative of equation (3.10), we obtain

$$\begin{aligned} d_{00}\ddot{\phi}_0 + [\dot{c}_{0\nu} + \dot{d}_{0\nu} + r_{0\nu} - d_{0i}(1/m_i)d_{i\nu}]\dot{\phi}_\nu \\ + [(\ddot{c}_{0\nu} + \dot{r}_{0\nu}) - d_{0i}(1/m_i)(\dot{c}_{i\nu} + r_{i\nu})]\phi_\nu = 0. \end{aligned} \tag{3.11}$$

Here we have substituted (3.9) into (3.11).

If $c_{00} \neq 0$, equation (3.11) yields no more constraints and becomes the true equation of motion. This corresponds to the case where no secondary constraints appear in the Dirac formulation and no difficulties occur. Thus, we assume that

$$c_{00} = 0 \tag{3.12}$$

from now on.

Substituting the condition (3.12) back into (3.11), we obtain

$$\begin{aligned} O_d\dot{\phi}_0 + [\dot{c}_{0j} + \dot{d}_{0j} + r_{0j} - d_{0i}(1/m_i)d_{ij}]\dot{\phi}_j \\ + [(\ddot{c}_{0\nu} + \dot{r}_{0\nu}) - d_{0i}(1/m_i)(\dot{c}_{i\nu} + r_{i\nu})]\phi_\nu = 0. \end{aligned} \tag{3.13}$$

Here we have defined $O_d$ as

$$O_d \equiv r_{00} - d_{0i}(1/m_i)d_{i0}. \tag{3.14}$$

It is important to notice that in (3.13) both $\phi_0$ and $\phi_j$ have the same order of time derivatives.

Assuming the invertibility condition that

$$O_d \neq 0 \tag{3.15}$$

we obtain from (3.9) and (3.13) the true equations of motion:

$$m_i\ddot{\phi}_i+[M_{ik}d_{kj}-d_{i0}O_d^{-1}(\dot{c}_{0j}+\dot{d}_{0j}+r_{0j})]\phi_j$$
$$+[M_{ik}(\dot{c}_{k\nu}+r_{k\nu})-d_{i0}O_d^{-1}(\ddot{c}_{0\nu}+\dot{r}_{0\nu})]\phi_\nu=0. \tag{3.16}$$

Here we have introduced the operator $M_{ik}$ defined by

$$M_{ik}=\delta_{ik}+d_{i0}O_d^{-1}d_{0k}(1/m_k). \tag{3.17}$$

This operator satisfies the relationship:

$$M_{ij}M_{jk}=M_{ik}+d_{i0}O_d^{-1}r_{00}O_d^{-1}d_{0k}(1/m_k). \tag{3.18}$$

Another true equation of motion comes from (3.13) by again taking the time derivative, and the result turns out to be

$$O_d\ddot{\phi}_0+\{2\dot{O}_d-d_{0i}(1/m_i)(-\dot{\bar{c}}_{i0}+r_{i0})-[\dot{c}_{0j}+r_{0j}-d_{0i}(1/m_i)d_{ij}](1/m_j)d_{j0}\}\dot{\phi}_0$$
$$+\{\ddot{d}_{0k}+2[\ddot{c}_{0k}+\dot{r}_{0k}-\dot{d}_{0i}(1/m_i)d_{ik}-d_{0i}(1/m_i)\dot{d}_{ik}]$$
$$-d_{0i}(1/m_i)(-\dot{\bar{c}}_{ik}+r_{ik})-[\dot{c}_{0j}+r_{0j}-d_{0i}(1/m_i)d_{ij}](1/m_j)d_{jk}\}\dot{\phi}_k$$
$$+\{(\ddot{c}_{0\nu}+\ddot{r}_{0\nu})-\dot{d}_{0i}(1/m_i)(c_{i\nu}+r_{i\nu})-d_{0i}(1/m_i)(\ddot{c}_{i\nu}+\dot{r}_{i\nu})$$
$$-[\dot{c}_{0j}+d_{0j}+r_{0j}-d_{0i}(1/m_i)d_{ij}](1/m_j)(\dot{c}_{j\nu}+r_{j\nu})\}\phi_\nu=0. \tag{3.19}$$

Here we have substituted (3.9) into (3.19) in the course of the derivation.

Thus, we have obtained the true equations of motion for the coordinates $\phi_i$ and $\phi_0$ (see equations (3.16) and (3.19)).

*3.1.1. Propagation.* The principal parts of the true equations of motion (3.16) and (3.19) become

$$m_i\delta_{ij}\ddot{\phi}_j+\ldots=0 \tag{3.20}$$
$$O_d\ddot{\phi}_0+\ldots=0. \tag{3.21}$$

Returning to the actual field theory model for which this mechanical model is an analogue, the fields $\phi_j$ have, by analogy, principal parts for the true equations of motion that are

$$m_i\delta_{ij}\Box\ddot{\phi}_j+\ldots=0 \tag{3.20a}$$
$$O_d\Box\phi_0+\ldots=0. \tag{3.21a}$$

It is now possible to consider the normals $n_\mu$ to the characteristic surfaces by replacing

$$\partial_\mu\to n_\mu \tag{3.22}$$

in the principal parts of the true equations of motion (3.20*a*) and (3.21*a*). If we now go to the specific frame

$$n_\mu=(n_0,0,0,0) \tag{3.23}$$

then the principal parts of both sets of equations of motion (field theory as well as mechanical model) (3.20) and (3.21), and (3.20*a*) and (3.21*a*) agree. Thus, in either case (for this particular frame) the characteristic determinant turns out to be

$$\mathscr{G}(n_0)\propto n_0^{2(N+1)}O_d\prod_{i=1}^{N}m_i. \tag{3.24}$$

*3.1.2. Quantisation.* If we assume that

$$r_{00} \neq 0 \tag{3.25}$$

equation (3.10) yields

$$\begin{aligned}\phi_0 &= -r_{00}^{-1}[d_{0j}\phi_j + (\dot{c}_{0j} + r_{0j})\phi_j] \\ &= -r_{00}^{-1} d_{0j}\phi_j + \dots\end{aligned} \tag{3.26}$$

To obtain the kinetic-energy part of the Lagrangian (3.3) we substitute (3.26) back into it. The result turns out to be

$$L = \dot{\phi}_i^* \mathcal{M}_{ij} \dot{\phi}_j + \dots \tag{3.27}$$

where $\mathcal{M}_{ij}$ is given by

$$\mathcal{M}_{ij} = m_i(\delta_{ij} - (1/m_i) d_{i0} r_{00}^{-1} d_{0j}). \tag{3.28}$$

Here $\mathcal{M}_{ij}$ is called the 'effective mass' [6].

Let $\pi_j^*$ be the canonical momenta conjugate to the coordinates $\phi_j$; then

$$\pi_j^* \equiv \partial L/\partial \dot{\phi}_j = \dot{\phi}_i^* \mathcal{M}_{ij}. \tag{3.29}$$

Assuming the usual equal time commutators for the field operators,

$$[\phi_i(x), \pi_j^*(y)] = \mathrm{i}\delta_{ij}\delta^{(3)}(x-y) \tag{3.30}$$

we arrive at

$$[\phi_i(x), \dot{\phi}_k^*(y)] = \mathrm{i}\delta^{(3)}(x-y)(1/m_i)M_{ik}. \tag{3.31}$$

Here we have used the useful relationships

$$\mathcal{M}_{ij}(1/m_j)M_{jk} = (1/m_i)M_{ij}\mathcal{M}_{jk} = \delta_{ik}. \tag{3.32}$$

## *3.2. Case (b)*

Discussions similar to those in case ($a$) apply here. Assuming the condition that

$$c_{ij} = 0 \tag{3.33}$$

we obtain the true equations of motion:

$$\begin{aligned}&m_0\ddot{\phi}_0 + [Md_{00} - d_{0i}O_{dij}^{-1}(\dot{c}_{j0} + \dot{d}_{j0} + r_{j0})]\dot{\phi}_0 \\ &\qquad + [M(\dot{c}_{0\nu} + r_{0\nu}) - d_{0i}O_{dij}^{-1}(\ddot{c}_{j\nu} + \dot{r}_{j\nu})]\phi_\nu = 0\end{aligned} \tag{3.34}$$

$$\begin{aligned}&O_{dij}\ddot{\phi}_j + \{2\dot{O}_{dij} - d_{i0}(1/m_0)(-\dot{c}_{0j} + r_{0j}) - [\dot{c}_{i0} + r_{i0} - d_{i0}(1/m_0)d_{00}] - (1/m_0)d_{0j}\}\dot{\phi}_j \\ &\qquad + \{\ddot{d}_{i0} + 2[\ddot{c}_{i0} + \dot{r}_{i0} - \dot{d}_{i0}(1/m_0)d_{00} - d_{i0}(1/m_0)\dot{d}_{00}] - d_{i0}(1/m_0)(-\dot{c}_{00} + r_{00}) \\ &\qquad - [\dot{c}_{i0} + r_{i0} - d_{i0}(1/m_0)d_{00}](1/m_0)d_{00}\}\dot{\phi}_0 \\ &\qquad + \{(\ddot{c}_{i\nu} + \ddot{r}_{i\nu}) - \dot{d}_{i0}(1/m_0)(\dot{c}_{0\nu} + r_{0\nu}) - d_{i0}(1/m_0)(\ddot{c}_{0\nu} + \dot{r}_{0\nu}) \\ &\qquad - [\dot{c}_{i0} + \dot{d}_{i0} + r_{i0} - d_{i0}(1/m_0)d_{00}](1/m_0)(\dot{c}_{0\nu} + r_{0\nu})\}\phi_\nu = 0\end{aligned} \tag{3.35}$$

where the following notation has been used:

$$O_{dij} = r_{ij} - d_{i0}(1/m_0)d_{0j} \tag{3.36}$$

$$M = \delta_{00} + d_{0i}O_{dij}^{-1}d_{j0}(1/m_0) \tag{3.37}$$

with

$$MM = M + d_{0i}O_{dij}^{-1}r_{jk}O_{dkl}^{-1}d_{l0}(1/m_0). \tag{3.38}$$

Here we have assumed the invertibility condition that

$$O_{dij} \neq 0. \tag{3.39}$$

*3.2.1. Propagation.* The characteristic determinant in the frame (3.23) becomes

$$\mathscr{G}(n_0) \propto n_0^{2(N+1)} m_0 \det O_{dij}. \tag{3.40}$$

*3.2.2. Quantisation.* Assuming that

$$r_{ij} \neq 0 \tag{3.41}$$

we obtain

$$\begin{aligned} \phi_i &= -r_{ij}^{-1}[d_{j0}\dot{\phi}_0 + (\dot{c}_{j0} + r_{j0})\phi_0] \\ &= -r_{ij}^{-1} d_{j0}\dot{\phi}_0 + \ldots . \end{aligned} \tag{3.42}$$

Substituting (3.42) back into the Lagrangian (3.3), we obtain the kinetic-energy part:

$$L = \dot{\phi}_0^* \mathscr{M} \dot{\phi}_0 + \ldots \tag{3.43}$$

where the effective mass is given by

$$\mathscr{M} = m_0[1 - (1/m_0) d_{0i} r_{ij}^{-1} d_{j0}]. \tag{3.44}$$

Assuming the usual equal time commutator that

$$[\phi_0(x), \pi_0^*(y)] = \mathrm{i}\delta^{(3)}(x-y) \tag{3.45}$$

we arrive at

$$[\phi_0(x), \dot{\phi}_0^*(y)] = \mathrm{i}\delta^{(3)}(x-y)(1/m_0)M \tag{3.46}$$

where

$$\pi_0^* = \partial L/\partial\dot{\phi}_0 = \dot{\phi}^* \mathscr{M}. \tag{3.47}$$

Here we have used the relationships

$$\mathscr{M}(1/m_0)M = (1/m_0)M\mathscr{M} = 1. \tag{3.48}$$

We have applied the Hamiltonian formalism proposed by Takahashi [6, 7]. The results we have obtained for the systems are equivalent to those of Dirac, who studied constrained Hamiltonian systems in general [9]. We underline the simplicity, generality and transparency of the above derivation.

The invertibility conditions (3.15) and (3.39), namely

$$O_d \neq 0 \tag{3.49}$$

are responsible for both the propagation and quantisation of the coordinates $\phi_\mu$. Thus we have arrived at the conclusion that the anomalies, if they happen, have a common origin inherent in the constraints of the systems.

## 4. Conclusions

The connection between the mechanical model and the field theory is complete if we replace the $\delta_{\mu\nu}$ in the mechanical model by $g_{\mu\nu}$ and, correspondingly, $m_k$ by $-m_k = -a_1$. In this case the operators in the field theory corresponding to $O_d$ are

$$(a) \qquad O_d = m^2 \tag{4.1}$$

$$(b) \qquad O_{dij} = m^2[g_{ij} + \mathrm{i}e(1-\alpha_{\mathrm{P}})(1/m^2)F_{ij}]. \tag{4.2}$$

Here we have used the condition (A1.21) from appendix 1. If $\alpha_P \neq 1$ the invertibility condition may not be satisfied by $O_{dij}$ since this matrix may become singular on a world sheet given by

$$\det O_{dij} = -[1 - e^2(1-\alpha_P)^2(1/m^4)\boldsymbol{H}^2] = 0. \tag{4.3}$$

This creates all the difficulties discussed so far.

If we let $L^{(c)}_{\mu\nu}(c = a, b)$ be the coefficient matrix for the principal parts of the true equations of motion in cases $(a)$ and $(b)$ (see equations (A2.6) and (A2.7) of appendix 2) we find (equations (A2.10) and (A2.11)) the following results in the frame $n_\mu = (n_0, 0, 0, 0)$:

$$(a) \qquad n_0^2 O_d = -L^{(a)}_{00}(n_0) \qquad L^{(a)}_{i0}(n_0) = 0 \tag{4.4}$$

$$(b) \qquad n_0^2 O_{dij} = -L^{(b)}_{ij}(n_0) \qquad L^{(b)}_{0j}(n_0) = 0. \tag{4.5}$$

This shows that the operators $O_d$ are responsible for the non-causal modes of propagation if they occur.

Correspondingly, we find that the field commutators are given by

$$[\phi_i^{(i)}(x), \dot{\phi}_j^{(1)*}(y)] = \mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})(g_{ij} + D_i O_d^{-1} D_j) \tag{4.6}$$

where

$$O_d = m^2$$

and

$$[\phi^{(0)}(x), \dot{\phi}^{(0)*}(y)] = \mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})(1 + D_i O_{dij}^{-1} D_j) \qquad \text{for } \alpha_P \neq 1. \tag{4.7}$$

Thus in either case the singularity of $O_d$ is responsible for both difficulties, i.e. if $O_d$ can vanish it can change sign.

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## Appendix 1. Classification of the first-rank tensor field theories

Consider a general massive tensor field $\phi$ of rank one and let $I$ be the identity operator so that

$$\phi = I\phi. \tag{A1.1}$$

Let

$$\mathscr{L} = \phi^* \Lambda(\partial)\phi \tag{A1.2}$$

be the assumed Lagrangian for such a free rank-one tensor field. The most general quadratic Lorentz covariant candidate for $\Lambda(\partial)$ is

$$\Lambda_{\mu\nu}(\partial) = -(a_1\partial^2 + a_2 m^2)g_{\mu\nu} - p_1\partial_\mu\partial_\nu \tag{A1.3}$$

where $a_1$, $a_2$, $p_1$ are three real parameters. The most general candidate, of degree four in derivatives, for a Klein-Gordon divisor is

$$m^4 D_{\mu\nu}(\partial) = (A_1\partial^4 + A_2 m^2\partial^2 + A_3 m^4)g_{\mu\nu} + (P_1\partial^2 + P_2 m^2)\partial_\mu \partial_\nu \tag{A1.4}$$

where $A_i (i=1,2,3)$, $P_j (j=1,2)$ are five real parameters.

We now impose the condition that $D(\partial)$ is the Klein-Gordon divisor for $\Lambda(\partial)$, namely

$$\Lambda(\partial)D(\partial) = D(\partial)\Lambda(\partial) = -(\partial^2 + m^2)I. \tag{A1.5}$$

Writing out (A1.5), we obtain, after rather tedious but straightforward calculations, seven independent equations for the eight parameters in $\Lambda(\partial)$ and $D(\partial)$. Next we solve these equations. Of the seven relationships, six are available for determining the five coefficients in the Klein-Gordon divisor. Of these, four are used to obtain the parameters $A_i$ $(i=1,2,3)$, and the remaining two are used to determine the parameters $P_j$ $(j=1,2)$. The remaining one then becomes a consistency condition. The results turn out to be

$$A_1 = 0 \tag{A1.6}$$

$$A_2 = (1 - a_1/p)(1/p) \qquad (a_1 = 0 \text{ or } p) \tag{A1.7}$$

$$A_3 = 1/p \tag{A1.8}$$

$$P_1 = -\chi(1-\chi) \tag{A1.9}$$

$$P_2 = [1 - (1 - a_1/p)](1/p) - \chi/p \tag{A1.10}$$

where $\chi$ is given by (2.20) and $p$ is defined by

$$p \equiv a_2 \neq 0. \tag{A1.11}$$

Finally the consistency condition yields

$$-p\chi^2(1-\chi) = 0 \tag{A1.12}$$

namely

$$\chi = 0, 1. \tag{A1.13}$$

Let $O_I$ $(I = 1,2)$ be the spin projection operators:

$$O_i = [P^{(1)}, P^{(0)}] \tag{A1.14}$$

where

$$P^{(1)}_{\mu\rho} = g_{\mu\rho} - \partial_\mu \partial^{-2}\partial_\rho \tag{A1.15}$$

$$P^{(0)}_{\mu\rho} = \partial_\mu \partial^{-2}\partial_\rho. \tag{A1.16}$$

These projection operators satisfy the relationships

$$\sum_i O_i = I \tag{A1.17}$$

$$O_i O_j = \delta_{ij} O_j \qquad (i,j = 1,2). \tag{A1.18}$$

It is convenient to rewrite the Klein-Gordon divisor in the form

$$D_{\mu\nu}(\partial) = [1-(1-a_1/p)](1/p)\tilde{P}^{(1)}_{\mu\nu} + \chi(1/p)\tilde{P}^{(0)}_{\mu\nu} + (1/m^2)(\partial^2 + m^2)(1-a_1/p)(1/p)g_{\mu\nu}. \tag{A1.19}$$

Here the tilde means the replacement of the inverse D'Alembertian operators $\partial^{-2}$ with $-m^{-2}$ in the relevant quantity. In the Takahashi-Umezawa formulation [8], the Klein-Gordon divisor satisfies the idempotency condition:

$$DD = D. \tag{A1.20}$$

This yields

$$p = 1. \tag{A1.21}$$

We now classify the first-rank tensor field theories in terms of the subsidiary conditions $\chi = 0, 1$.

*A1.1. $\chi = 0$*

This corresponds to having the Lorentz condition satisfied by the field.

*A1.1.1. $a_l = p$; $p_l = -p$.* Pure spin-1:

$$D = \tilde{P}^{(1)} \tag{A1.22}$$

$$\Lambda_{\mu\nu} = -(\partial^2 + m^2) g_{\mu\nu} + \partial_\mu \partial_\nu \equiv \Lambda^{(L)}_{\mu\nu}. \tag{A1.23}$$

This is the famous Proca theory [10].

*A1.1.2. $a_l = 0$; $p_l = 0$.* Does not yield any theory:

$$D = (1/m^2)(\partial^2 + m^2) I \tag{A1.24}$$

$$\Lambda_{\mu\nu} = -m^2 g_{\mu\nu}. \tag{A1.25}$$

*A1.2. $\chi = 1$*

*A1.2.1. $a_l = p$; $p_l = 0$.* Mixed spin-1 and 0:

$$D = I \tag{A1.26}$$

$$\Lambda_{\mu\nu} = -(\partial^2 + m^2) g_{\mu\nu}. \tag{A1.27}$$

This case is reducible, since the reducibility condition

$$O_1 \Lambda(\partial) O_2 + O_2 \Lambda(\partial) O_1 = 0 \tag{A1.28}$$

holds.

*A1.2.2. $a_l = 0$; $p_l = p$.* Pure spin-0:

$$D_{\mu\nu} = \tilde{P}^{(0)}_{\mu\nu} + (1/m^2)(\partial^2 + m^2) g_{\mu\nu} \tag{A1.29}$$

$$\Lambda_{\mu\nu} = -m^2 g_{\mu\nu} - \partial_\mu \partial_\nu \equiv \Lambda^{(B)}_{\mu\nu}. \tag{A1.30}$$

This corresponds to having the field satisfying the Bianchi identity.

Thus, only two theories, A1.1.1 and A1.2.2, survive, as is to be expected. It is remarkable that there exists a reciprocal relationship [11]:

$$\Lambda^{(L)}_{\mu\rho}(1/m^2)\Lambda^{(B)\rho}{}_{\nu} = \Lambda^{(B)}_{\mu\rho}(1/m^2)\Lambda^{(L)\rho}{}_{\nu} = -(\partial^2 + m^2) g_{\mu\nu}. \tag{A1.31}$$

This shows that the two operators $\Lambda^{(L)}$ and $\Lambda^{(B)}$ are, up to a factor of $1/m^2$, each other's Klein-Gordon divisor. Therefore there is a simple equivalence theorem between the Lorentz condition and the Bianchi identity, as has been discussed previously [11, 12].

**Appendix 2. The characteristic determinants**

The Euler–Lagrange equation of motion follows from equation (2.1), namely

$$[\Lambda_{\mu\rho}(D)+\Lambda^{(\mathrm{P})}_{\mu\rho}]\phi^\rho=0. \tag{A2.1}$$

Written out, this becomes

$$[\Lambda^{(\mathrm{L})}_{\mu\rho}(D)+\Lambda^{(\mathrm{P})}_{\mu\rho}]\phi^\rho=\{-(D^2+m^2)g_{\mu\rho}+g_{\mu\alpha}[\alpha_{\mathrm{P}}D^\alpha D_\beta+(1-\alpha_{\mathrm{P}})D_\beta D^\alpha]g^\beta{}_\rho\}\phi^\rho=0 \tag{A2.2}$$

$$[\Lambda^{(\mathrm{B})}_{\mu\rho}(D)+\Lambda^{(\mathrm{P})}_{\mu\rho}]\phi^\rho=\{-m^2g_{\mu\rho}-g_{\mu\alpha}[\alpha_{\mathrm{P}}D^\alpha D_\beta+(1-\alpha_{\mathrm{P}})D_\beta D^\alpha]g^\beta{}_\rho\}\phi^\rho=0. \tag{A2.3}$$

Here we have used the conditions A1.1.1 for (A2.2) and the conditions A1.2.2 for (A2.3) together with (A1.23) as well as (3.6).

Contracting (A2.2) with $D^\mu$, we obtain

$$D_\rho\phi^\rho=\mathrm{i}e(1/m^2)(F_{\rho\alpha}D^\alpha-\alpha_{\mathrm{P}}D^\alpha F_{\alpha\rho})\phi^\rho. \tag{A2.4}$$

The Euler–Lagrange equation of motion (A2.3) yields

$$D_\lambda\phi_\mu-D_\mu\phi_\lambda=\mathrm{i}e(1/m^2)[F_{\lambda\mu}D_\rho-(1-\alpha_{\mathrm{P}})(D_\lambda F_{\mu\rho}-D_\mu F_{\lambda\rho})]\phi^\rho. \tag{A2.5}$$

In the limit that $e$ tends to zero, equations (A2.4) and (A2.5) become the Lorentz condition and the Bianchi identity, respectively.

The true equations of motion are obtained by substituting (A2.4) back into (A2.2) and by contracting (A2.5) with $D^\lambda$ and using (A2.3). The results turn out to be

$$[-(D^2+m^2)g_{\mu\rho}+\mathrm{i}e(1/m^2)D_\mu(F_{\rho\alpha}D^\alpha-\alpha_{\mathrm{P}}D^\alpha F_{\alpha\rho})+(1-\alpha_{\mathrm{P}})\mathrm{i}eF_{\mu\rho}]\phi^\rho=0 \tag{A2.6}$$

$$[-(D^2+m^2)g_{\mu\rho}+\mathrm{i}e(1/m^2)[D^\alpha F_{\alpha\mu}D_\rho-(1-\alpha_{\mathrm{P}})(D^2F_{\mu\rho}-D^\alpha D_\mu F_{\alpha\rho})] +\alpha_{\mathrm{P}}\mathrm{i}eF_{\mu\rho}]\phi^\rho=0. \tag{A2.7}$$

To find the normals $n_\mu$ to the characteristic surfaces, we replace

$$\partial_\mu\to n_\mu \tag{A2.8}$$

in the principal parts of the true equations of motion, equations (A2.6) and (A2.7), and calculate the determinant $\mathscr{G}(n)$ of the resulting coefficient matrix. Here $\mathscr{G}(n)$ is called the characteristic determinant. The results which we have obtained are

$$\mathscr{G}^{(a)}(n)=\det L^{(a)}_{\mu\rho}(n) \qquad a=\mathrm{L}\text{ and }\mathrm{B} \tag{A2.9}$$

where

$$L^{(\mathrm{L})}_{\mu\rho}(n)=-n^2g_{\mu\rho}-\mathrm{i}e(1/m^2)n_\mu n^\alpha(1+\alpha_{\mathrm{P}})F_{\alpha\rho} \tag{A2.10}$$

and

$$L^{(\mathrm{B})}_{\mu\rho}(n)=-n^2g_{\mu\rho}+\mathrm{i}e(1/m^2)[n^\alpha F_{\alpha\mu}n_\rho-(1-\alpha_{\mathrm{P}})(n^2F_{\mu\rho}-n^\alpha n_\mu F_{\alpha\rho})]. \tag{A2.11}$$

To avoid cumbersome computations we take the special frame

$$n_\mu=(n_0,0,0,0). \tag{A2.12}$$

This frame plays a crucial role for the true equations of motion in non-covariant form. A rather tedious computation of the determinants then yields

$$\mathscr{G}^{(\mathrm{L})}(n_0)=(n_0^2)^4 \tag{A2.13}$$

$$\mathscr{G}^{(\mathrm{B})}(n_0)=(n_0^2)^4[1-e^2(1-\alpha_{\mathrm{P}})^2(1/m^4)\boldsymbol{H}^2] \tag{A2.14}$$

where $H^1 = F_{23}$, $H^2 = F_{31}$ and $H^3 = F_{12}$. In covariant form, equations (A2.13) and (A2.14) turn out to be

$$\mathscr{G}^{(\mathrm{L})}(n) = (n^2)^4 \tag{A2.15}$$

$$\mathscr{G}^{(\mathrm{B})}(n) = (n^2)^3[n^2 + (1/m^4)e^2(1-\alpha_\mathrm{P})^2(nF^\mathrm{d})^2] \tag{A2.16}$$

where $F^\mathrm{d}_{\mu\nu}$ is the dual field of $F_{\mu\nu}$ and is defined by

$$F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$$

with the convention $\varepsilon^{0123} = 1$.

The nature of propagation follows from the characteristic roots defined by

$$\mathscr{G}(n) = 0. \tag{A2.17}$$

If we choose $\alpha_\mathrm{P} = 1$, then (A2.15) and (A2.16) yield

$$\mathscr{G}^{(a)}(n) = (n^2)^4 = 0 \qquad \text{for } a = \mathrm{L} \text{ and } \mathrm{B}. \tag{A2.18}$$

Thus, we have the roots $n_\mu$ to be

$$n_0 = \pm|n|. \tag{A2.19}$$

Since the roots are real and on the light cones, the equations of motion are hyperbolic and the propagation is causal.

For the Proca field, the theory is independent of the Pauli term, while for the spin-0 field we have to choose

$$\alpha_\mathrm{P} = 1 \tag{A2.20}$$

otherwise non-causal modes of propagation arise in that theory.

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